Introduction to Quantum Algorithms-Lecture 4

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Problem

Fron a list of $q = 2^n$ items to peak the k-th item.
Example from a phone book with 10^6 names to find the k-th name.

Mathematical Introduction

Let $|\sigma\rangle$ and $|s\rangle$ two perpendicular unity vectors

$$< \sigma|s >=< s|\sigma >= 0, \; < \sigma|\sigma >=< s|s >= 1, |\tau >= \sin\omega|\sigma\rangle + \cos\omega|s\rangle$$

Matrix representation

$$|\sigma\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |s\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; |\tau\rangle \leftrightarrow \begin{pmatrix} \sin\omega \\ \cos\omega \end{pmatrix},$$

Grover or rotation operator

$$G = (2|\tau\rangle\langle\tau| - \mathbb{I})(\mathbb{I} - 2|\sigma\rangle\langle\sigma|) \leftrightarrow \begin{pmatrix} \cos 2\omega & \sin 2\omega \\ -\sin 2\omega & \cos 2\omega \end{pmatrix},$$

$$G^m \leftrightarrow \begin{pmatrix} \cos 2m\omega & \sin 2m\omega \\ -\sin 2m\omega & \cos 2m\omega \end{pmatrix},$$

$$|\rho\rangle = \sin\theta|\sigma\rangle + \cos\theta|s\rangle \rightsquigarrow G^m|\rho\rangle \leftrightarrow \begin{pmatrix} \cos(2\omega + \theta) \\ \sin(2\omega + \theta) \end{pmatrix}$$

$$|\sigma\rangle = |\bar{k}>, \ |\tau\rangle = \frac{1}{\sqrt{q}} \sum_{\ell=0}^{q-1} |\bar{\ell}> \rightsquigarrow |s\rangle = \frac{1}{\sqrt{q-1}} \sum_{\ell=0, \ell \neq k}^{q-1} |\bar{\ell}>$$

$$|\tau\rangle = \frac{1}{\sqrt{q}}|\sigma\rangle + \sqrt{\frac{q-1}{q}}|s\rangle \rightsquigarrow \sin\omega = \frac{1}{\sqrt{q}}$$

$$|\rho\rangle = |\tau\rangle = \sin\omega|\sigma\rangle + \cos\omega|s\rangle$$

$$G^m|\rho\rangle = \sin\left((2m+1)\omega\right)|\sigma\rangle + \cos\left((2m+1)\omega\right)|s\rangle$$

Approximation

If $(2m+1)\omega \sim \frac{\pi}{2}$ then $G^m|\rho\rangle \sim |\sigma\rangle$

$$\sin\omega = \frac{1}{\sqrt{q}} \rightsquigarrow \omega \sim \frac{1}{\sqrt{q}} \Rightarrow (2m+1)\frac{1}{\sqrt{q}} \sim \frac{\pi}{2} \rightsquigarrow m \sim \frac{\pi}{2}\sqrt{q}$$

n	$q = 2^n$	m	Prob
10	1024	24	99.85%
20	10^6	803	99.999%

Grover's Algorithm

Step 1 : We start with the initial state $|\text{in}\rangle = |0\rangle = |00\ldots 0\rangle$

Step 2 : We apply the operator $H^{\otimes n}$ on the initial state and we get the state $|\rho\rangle = \frac{1}{\sqrt{q}} \sum_{\ell=0}^{q-1} \left|\bar{\ell}\right\rangle$

Step 3 : We apply $m = \left[\left(\frac{\pi}{2\omega} - 1\right)/2\right]$ times the Grover operator, with $\omega = \arcsin \frac{1}{\sqrt{q}}$

$$G = (2|\tau\rangle\langle\tau| - \mathbb{I})\,(\mathbb{I} - 2|\sigma\rangle\langle\sigma|) \leftrightarrow \begin{pmatrix} \cos 2\omega & \sin 2\omega \\ -\sin 2\omega & \cos 2\omega \end{pmatrix},$$

The final state is given by the formula

$$|\text{fin}\rangle G^m|\rho\rangle = \sin\left((2m+1)\omega\right)|\sigma\rangle + \cos\left((2m+1)\omega\right)|s\rangle$$

Step 4 : The probality to be in the final state $|\sigma\rangle$ is equal to $\sin^2((2m+1)\omega$